РОЗДІЛ 2. СПРЯМОВАНІСТЬ НАВЧАННЯ ДИСЦИПЛІН ПРИРОДНИЧО-МАТЕМАТИЧНОГО ЦИКЛУ НА РОЗВИТОК ІНТЕЛЕКТУАЛЬНИХ УМІНЬ ТА ТВОРЧИХ ЗДІБНОСТЕЙ УЧНІВ ТА СТУДЕНТІВ

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GEOMERY PROBLEMS WITH AN IMPLICITLY DEFINED STRUCTURE – A MEANS FOR DEVELOPING INTELLECTUAL SKILLS IN LEARNERS

Стаття присвячена планіметричним задачам, пов'язаним із серединою трикутника та вписаним у нього колом. Важливою проблемою в цих проблемах є те, що для визначення їхньої структури потрібні більш конкретні, додаткові міркування, оскільки вона не визначена явно. По одній проблемі, яка називається «основною», зроблена детальна методична розробка, яка має дві мети. Перше завдання – комплексно окреслити їх «освітні» рішення, спрямовані на цілеспрямовану підготовку учнів, які складатимуть атестат зрілості з математики чи кандидатський іспит, тому ця проблема має критеріальний характер. Інша мета — продемонструвати методичну технологію створення серії нових проблем, які випливають із цієї основної проблеми. Розглянуто різні ідеї створення систем задач і наведено відповідні формулювання.

Ключові слова: *методична розробка, структура задачі, способи розв'язування, інтелектуальні вміння, складання задач.*

Problem statement. According to the national strategy for comprehensive development of students' personality, the main focus of education is their intellectual development. A. Krasteva notes that "intellect is definitely associated with intellectual actions, which in itself is an indicator of the development of a person's mental activity, i.e. the procedural characteristics of intelligence are determined by a person's ability to solve problems – reasoning ability" [6, p. 114]. This fully applies to learning mathematics, which is impossible without solving problems. When performing this activity, students not only carry out relevant constructions, transformations and learn the formulations of definitions, theorems, rules, but also learn to express their thoughts and ideas clearly, acquire purposeful intellectual skills to reason correctly, compare and contrast facts, to discover the common and the different in them, to make correct conclusions, to think critically, to argue their conclusions, to master methods of acquiring knowledge. According to D. Polya "Problem-solving is a specific feature of the intellect, and intelligence is a special gift of man, so problem-solving can be considered as one of the most characteristic manifestations of human activity. To master mathematics means to be able to solve problems, not only standard but also those that require originality, ingenuity, independence of thinking" [15, p. 5]. Therefore, the activity of solving mathematical problems is quite complicated, and its description is relatively complex and time-consuming. That is why this activity includes many actions and operations, such as analyzing the text of the problem, building auxiliary models, reformulating the sought (thesis), the given, comparing with familiar problems, etc. This can explain the fact that well-known authors of textbooks and teaching aids prefer, with a view to teaching students, to describe *the process of searching for and discovering a solution* rather than directly presenting the solution itself, for example: [2], [12], [16], [20].

Analysis of current research. In order for the training to be successfully implemented, it is necessary for both the trainers and the trainees to master and apply the correct methodology for carrying out the activity of solving mathematical problems. For this purpose, it is appropriate that current teachers, as well as students preparing to become secondary school mathematics teachers, should be able to carry out methodical development of the mathematical problems included in the textbooks, books with problems and teaching aids.

Methodical development of a mathematical problem in this article is understood as the implementation of the four-stage scheme for solving problems proposed by D. Polya [14, p. 72], which has been repeatedly quoted, applied, enriched and improved by a number of methodologysts K. Slavov [19], R. Cherkasov & A. Stolyar [1], V. Krupich [7], I. Ganchev [3], P. Petrov [13], etc.

Research shows that the activity of solving mathematical problems has a staged character, that is why some authors consider it as a "solving process" [18, p. 170].

The model of D. Poya [14] is extensively enriched in the article [9, p. 96 – 100], where a modern version of technological model of the activity of solving mathematical problems (ASMP) is developed, which has a large practical application. It not only describes in detail the main stages and their corresponding sub-stages of the ASMP, but also presents a diagram of this activity, which also reflects the connections between individual stages and sub-stages. In such a way, it presents both the content and the very structure of the activity of solving mathematical problems, that is why it is called a model of ASMP. In this regard, K. Koleva notes that "the stages of this model can be related to the corresponding stages of the psychological concept of the formation of skills" [5, p. 12].

In the monograph [10], the ASMP model has been repeatedly applied to solving multiple problems that have different structures or are from different areas of mathematics, but the question of composing new problems has not been considered there.

The objectives of the article is to consider various ideas for creating problem systems and provide appropriate formulations**.**

Findings. Educational pedagogical practice and teaching experience show that the training of students, which is aimed at mastering the ASMP and its model, contributes to the assimilation and deepening not only of knowledge about the theory and structure of mathematical problems, but also to the acquisition of generalized mathematical knowledge and intellectual skills at a reflective level. This confirms the thesis that the problem of the methodical development of tasks continues to be relevant.

In this article, the ASMP model is applied to a planimetric problem (with an implicitly defined structure), which will be subjected to a detailed methodological development, including the sub-stage of composing new problemss by applying different ideas and methods. For definiteness, we call this output problem the "basic problem".

Basic problem. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio $\frac{r}{R}$ of the radii r and R, respectively, of the circle inscribed in ΔABC and the circle circumscribed around it.[1]

Solution. ***I stage***: Learning the statement and the requirements of the problem and determining its structure. Each right triangle is uniquely defined by two parameters, at least one of which must be metric. No new parameters are needed for the inscribed and circumscribed circles because, according to the relevant theorems, their centers are uniquely determined for each triangle. There is no metric element given in the statement of the problem, and the non-metric element is implicitly set here as well. To make it easier to understand the statement of the problem and state the non-metric element in it, it is useful to introduce the following notations: let CM be the median to the hypotenuse AB, and let O be the center of the inscribed circle k in ΔABC. Since the midcenter G is given to lie on the inscribed circle, and by definition it is the point of intersection of the medians, then G is the common point of the median CM and the circle k. But CM and k have two common points. Which one is the midcenter? Since it divides each median in the ratio 2:1 from the vertex, G must be the intersection of CM and k that is closer to M. It should be noted,

[1] This problem is a modification of problem M1224, which was proposed by V. Senderov [17, p. 26] for the Kvant magazine, 1990, vol. 5, p. 26 in the rubric "Quantum problem" and one of its variants is discussed in the article [4, p. 46]. Here, from the text of the cited problem M1224, essentially only the given is borrowed: "The medians of a right-angled triangle intersect at its inscribed circle. Find the angles of this triangle", and what is sought is formulated by the author of this article.

however, that the given right triangle ΔABC cannot be isosceles, because in that case one of the common points of CM and k is the point of contact of the circle k with the hypotenuse, and the other common point is closer to the vertex C, but the specified 2:1 ratio is not fulfilled for them. Therefore, the statement and the requirements given in the problem will be fulfilled when the right triangle ΔABC is not isosceles. Let us, for definiteness, assume, for example, that $AC < BC$, which is equivalent to the inequality $\measuredangle ABC < \measuredangle BAC$ (Fig. 1).

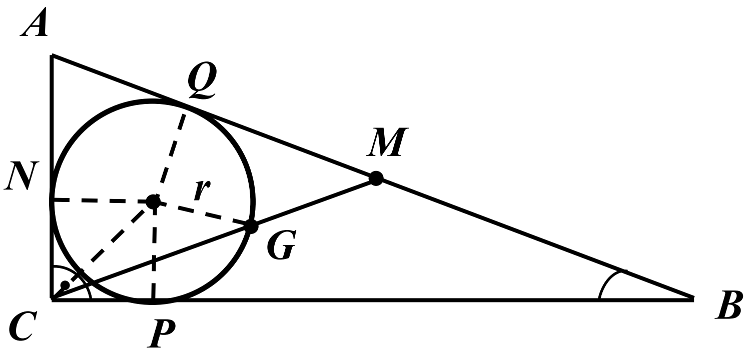

Fig. 1

In such a way, the implicitly assigned element in the problem is expressed (has a relationship) with the specified relation in which the intersection point G of CM and k divides the median CM. Therefore, it can be concluded that the figure is defined by the *similarity relation* – this is the structure of the basic problem and represents the main information in it. Everything else in the statement (both given and sought) refers to the specific information. There is no redundant information in the basic problem statement.

II stage: Generating a solution idea.

The main information in the problem (the figure is defined to the *similarity relation*) points to the idea: introducing an additional metric parameter. But which is suitable? Such a linear element would be suitable, which could make a connection between what is sought in the problem and the implicitly set ratio. Since the ratio $\frac{r}{R}$ is sought, either r, or R, or some other segment but related to both radii r and R could be chosen as an additional metric parameter.

III stage: Implementation of the idea.

Perhaps, taking into account that AB is the hypotenuse in ΔABC, and it is a diameter of the circumscribed circle and $CM = \frac{1}{2}AB = R$, and on the other hand, the $CG: GM = 2: 1$ is satisfied, it seems that the third possibility is preferable, namely – to choose the length x of the segment GM, $x > 0$, as a parameter. Then $CG = 2x$, $R = 3x$. It remains to express the radius r of the inscribed circle by x. For this purpose it (r) must be included in some triangle for which there is some information, i.e. for which sides or for some of them it is possible to express by the parameter x. Such could be those triangles in which both r (directly or indirectly) and a segment expressed by x are involved. In order to account the condition that the midcenter $G \in k$ and that the segment CG is already expressed by x, and the segment CO is the bisector of the right angle at the vertex C and at the same time it is very easily expressible by r (Why?), it would obtained a suitable triangle if the radius $OG = r$ is additionally constructed. This gives the triangle ΔCOG. For it, it is satisfied: $CO = r\sqrt{2}$ (CO is a diagonal in the square $CPON$), $OG = r$ and $CG = 2x$. Therefore, the desired relationship between r and x can be achieved by applying the cosine theorem for ΔCOG. However, the question arises here: "For which side should it be applied?". Since $\sphericalangle OCG$ is the angle between the bisector of the right angle ($\measuredangle ACB$) and the median CM, and it is expressible by $\measuredangle ABC = \beta$, it is appropriate to apply the cosine theorem to the side OG, which lies opposite $\sphericalangle OCG$. Since $\sphericalangle OCG = \sphericalangle OCB - \sphericalangle BCM$, and $\sphericalangle BCM = \sphericalangle CBM = \beta$, then $\sphericalangle OCG = 45^0 - \beta$. Then the equalities are satisfied:

$OG^2 = CO^2 + CG^2 - 2.CO.CG.\cos(45^0 - \beta)$

$r^2 = 2r^2 + 4x^2 - 2.r\sqrt{2}.2x.(\cos 45^0.\cos\beta + \sin 45^0.\sin\beta)$

$r^2 + 4x^2 - 4r\sqrt{2}.x\frac{\sqrt{2}}{2}(\cos\beta + \sin\beta) = 0$ (1)

In order to find $\cos\beta + \sin\beta$, the characteristic equality for a right triangle can be used: $a + b = 2r + 2R$, which, in order to obtain the sought ratio $\frac{r}{R}$, it is appropriate to be written in the following form: $\frac{a}{2R} + \frac{b}{2R} = \frac{r}{R} + 1$.

But $\frac{a}{2R} = \cos\beta$, a $\frac{b}{2R} = \sin\beta$.

Therefore $\cos\beta + \sin\beta = \frac{r}{R} + 1$ (2)

Then equality (1) takes the form:

$r^2 + 4x^2 - 4rx\left(\frac{r}{R} + 1\right) = 0$ (3)

In order to obtain a suitable equation for the sought ratio $\frac{r}{R}$, it is appropriate to divide equation (3) by R^2, resulting in

$\frac{r^2}{R^2} + \frac{4x^2}{R^2} - \frac{4rx}{R^2}\left(\frac{r}{R} + 1\right) = 0$ (4)

But $R = 3x$, therefore, it is appropriate to write (4) as follows

$\frac{r^2}{R^2} + \frac{4x^2}{9x^2} - 4\frac{r}{R}.\frac{x}{3x}\left(\frac{r}{R} + 1\right) = 0$ or, after simplification, in the form:

$\left(\frac{r}{R}\right)^2 + \frac{4}{9} - \frac{4}{3}\left(\frac{r}{R}\right)\left(\frac{r}{R} + 1\right) = 0$ (5)

It is appropriate to make the substitution here $\frac{r}{R} = t,\ t \in (0; 1)$, by which from (5) the equation is obtained $t^2 + \frac{4}{9} - \frac{4}{3}t(t + 1) = 0$. Simplifying, it takes the following normal quadratic equation form $3t^2 + 12t - 4 = 0$. The latter has only one positive root and it is $t = \frac{-6+4\sqrt{3}}{3} \in (0; 1)$. Hence the sought relation in the basic problem **2** has value: $\frac{r}{R} = \frac{-6+4\sqrt{3}}{3}$.

In order to implement the idea of I. B. Olbinskii for a "development of the problem" [11, p. 15], it is also appropriate to note that from equality (2), with the thus found value of the ratio $\frac{r}{R}$ for the angle β, the trigonometric equation $\mathrm{ccos}\,\beta + \sin\beta = \frac{-6+4\sqrt{3}}{3} + 1$ is obtained, which, after simplifying its right side, takes the form $\cos\beta + \sin\beta = \frac{4\sqrt{3}}{3} - 1$ (6).

Since this equation is of the type: $a.\sin\beta + b.\cos\beta = c$ and $c^2 \leq a^2 + b^2$ is fulfilled for its coefficients (even in the particular case the strict inequality in it is present, which is easily verified), then equation (6) will have a solution. Various ideas can be applied to find it. Since $c > 0$ is fulfilled for the free term c of equation (6) and the left side is also positive, there is no danger of obtaining foreign solutions when squaring its two sides. This exponentiation results in the following basic trigonometric equation $\sin 2\beta = \frac{8}{3}\left(2 - \sqrt{3}\right) \approx 0{,}72$. Since it was assumed above that $AC < BC$, i.e. $\measuredangle ABC < 45^o < \measuredangle BAC$, then $2\beta < 90^o$. Therefore, the last equation for 2β has a unique solution in the interval $((0^o;\ 90^o)$, i.e. $\beta \in (0^o;\ 45^o)$.

The comments made above can be continued along the following lines. Through the formula $\sin 2\beta = \frac{2.tg\beta}{1+tg^2\beta}$, from the equation $\sin 2\beta = \frac{8}{3}\left(2 - \sqrt{3}\right)$ a fractional equation for $tg\beta$ is obtained, namely $\frac{2.tg\beta}{1+tg^2\beta} = \frac{8}{3}\left(2 - \sqrt{3}\right)$. It leads to the following quadratic equation for $tg\beta$, which is reciprocal:

$8\left(2 - \sqrt{3}\right)tg^2\beta + 6.tg\beta - 8\left(2 - \sqrt{3}\right) = 0.$

If the senior coefficient and its free term are rationalized, the equation is obtained $4.tg^2\beta + 3\left(2 + \sqrt{3}\right)tg\beta - 4 = 0$. It has only one positive root for $tg\beta$, which, however, is expressed by a complex radical, namely $tg\beta = \frac{1}{8}\left(\sqrt{127 + 36\sqrt{3}} - 6 - 3\sqrt{3}\right)$.

The reasoning and research conducted here has a twofold role. On the one hand, they are a concrete implementation of the idea of "development" of the basic problem, and on the other hand, they can also support the creation of new problems during the implementation of the next stage. And all this contributes to the development of the intellect of the learners.

IV stage: A look back. Other ideas for solution can be presented here, their advantages and disadvantages clarified, and also compared in terms of rationality. Thus, for example, since the figure in the basic problem is defined to the similarity relation, it is possible to apply the method of similarity, i.e. to find the required ratio for one of the many similar triangles – for example, the one whose hypotenuse is 6 units. Then $R = 3$, $CG = 2$, $GM = 1$. The rest of the reasoning repeats those that were carried out during the application of the parameterization method, as a result of which, for the sought ratio $\frac{r}{R}$, we again arrive at equation (5), having one positive root: $\frac{r}{R} = \frac{-6+4\sqrt{3}}{3}$. Of course, in order to realize the idea of developing the problem, the reasoning concerning the angle β can also be carried out.

Another idea for finding a solution to the basic problem is based on the metric dependence that connects the tangent and the secant constructed from an external point to the given circle (Fig. 2).

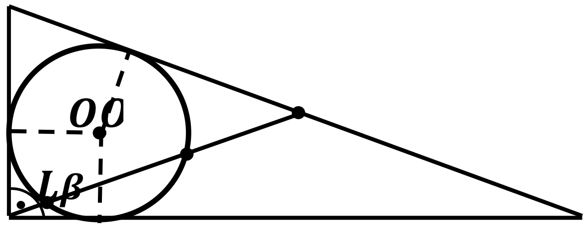

Fig. 2

It shows that the median CM, crossing the inscribed circle at points G and L, forms two secants. One is CG with outer part CL and the other is ML with outer part MG. On the other hand, CN is tangent to this circle and therefore the equality is satisfied

$CG.CL = CN^2$ (1)

Analogously for the tangent MQ we have the equality

$ML.MG = MQ^2$ (2)

In order to use these relations, since the considered figure in the basic problem is defined to similarity, it is necessary, as in the first solution, to introduce a metric parameter. Here it is also appropriate to accept the length x of the segment MG as such, because it directly expresses the lengths of the segments CG, CM, AB. Let's $MG = x$, $(x > 0)$. Therefore $CG = 2x$, $CM = 3x$, $AB = 6x$, $R = 3x$. Then, to find the required ratio $\frac{r}{R}$, it is sufficient to calculate the ratio $\frac{r}{x}$ (because $\frac{r}{R} = \frac{1}{3}.\frac{r}{x}$). Therefore, a relationship must be found between r and x. For this purpose, it must be taken into account that $CN = r$ and equality (1) $CG.CL = CN^2$ is fulfilled. In it the segment CL is not expressed by x, and since the chord LG is contained both in the secant CG and in the secant ML, it is convenient to temporarily accept the length of this chord as an additional parameter. Let's mark $LG = p$, where $0 < p < 2x$. Then $CL = 2x - p$ and therefore equality (1) yields the form

$2x(2x - p) = r^2$ (3)

Since $ML = x + p$, the equality (2) takes the following form

$(x + p)x = MQ^2$ (4)

It remains to be expressed by the parameters the tangent MQ. If $AC < BC$, i.e. $b < a$, then the point of contact Q of the inscribed circle with the hypotenuse AB is between A and M. Then $MQ = AM - AQ$. Since $AM = R$, and $AQ = AN = AC - CN$, then $AQ = b - r$. Therefore

$MQ = R - b + r$ (5)

Since $CPON$ is a square with side r, using the property of tangents from point C to the circle, we get that $r = CN = CP = CB - BP$. But $CB = a$ and $BP = BQ = BM + MQ = R + MQ$. Therefore

$r = a - R - MQ$ (6)

From (5) and (6) it follows that $MQ = R - b + a - R - MQ$, from which it follows that $MQ = \frac{a-b}{2}$. This allows equality (4) to be written in the form

$(x+p)x = \frac{(a-b)^2}{4}$ (7)

From equations (3) and (7) the following system is obtained

$$\begin{cases} 4x^2 - 2px = r^2 \\ 2x^2 + 2px = \frac{(a-b)^2}{2} \end{cases} \quad (8)$$

From it, by eliminating the parameter p, the following relation between r, x, a and b is obtained.

$6x^2 = r^2 + \frac{(a-b)^2}{2}$ (9).

To find the $\frac{r}{x}$ ratio, the a and b legs must be eliminated. For this purpose, the characteristic equalities for a right triangle are used:

$a + b = 2r + 2R$ (10) и $a^2 + b^2 = c^2 = 4R^2$ (11).

By squaring equality (10) and applying (11) is obtained the equality

$ab = 4rR + 2r^2$ (12)

The product ab can be expressed in terms of r and x in different ways. Thus, for example, if we transform equality (9) and apply (11), we get

$ab = 2R^2 + r^2 - 6x^2$ (13)

Since $R = 3x$, equalities (12) and (13) take the form, respectively:

$ab = 12rx + 2r^2$ and $ab = 12x^2 + r^2$.

From these, by equating the right sides and simplifying, a homogeneous quadratic equation with respect to r and x is obtained

$r^2 + 12rx - 12x^2 = 0$ (14)

Since the ratio $\frac{r}{x}$ is sought, it is appropriate to write the equation (14) in the following form

$\left(\frac{r}{x}\right)^2 + 12.\frac{r}{x} - 12 = 0.$

This quadratic equation has only one positive root, namely $\frac{r}{x} = -6 + 4\sqrt{3}$. Then for the required relation the following result is obtained:

$\frac{r}{R} = \frac{-6+4\sqrt{3}}{3}$.

As can be seen from the realization of these two ideas for solving the basic problem, a number of instructive conclusions can be drawn in a reflexive plan. On the one hand, the discovery of the presented "educational" solutions required the use of knowledge from different sections of the school mathematics course – both geometry, algebra and trigonometry. Discovering the very structure of the problem, which is implicitly defined in the statement, requires making appropriate additional constructions. As a result, different ideas arise for the application of specific mathematical knowledge and relevant skills that the problem solver should possess. On the other hand, the reasoning carried out and their presentation in formulating these solutions contribute significantly to the development of relevant intellectual skills in learners.

The Look Back stage also includes the creation of new problems. From the basic problem, new problems can be created in different aspects.

A. Problems having the same structure – the figures in them must be defined to the *similarity relation*. Here, the possibilities for varying what is sought, and even more so for changing what is given, are quite limited due to the specific condition: that the midcenter G of the given right triangle lies on the circle inscribed in it. Thus, by keeping what is given in the basic problem and varying what is sought, we formulate the following problems.

Problem A.1. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio $\frac{m_c}{r}$, where m_c is the median to the hypotenuse and r is the radius of the circle inscribed in ΔABC.

Problem A.2. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio $\frac{r}{AB}$, where r is the radius of the circle inscribed in ΔABC and AB is its hypotenuse.

Problem A.3. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio of the legs AC: BC, where AC is the smaller leg in ΔABC.

Problem A.4. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio $\frac{r}{AC}$, where r is the radius of the circle inscribed in ΔABC and AB is the smaller leg.

Problem A.5. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio $\frac{R}{BC}$, where R is the radius of the circle circumscribed around ΔABC and BC is its larger leg.

Problem A.6. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio of the perimeter of ΔABC and the radius of the circle inscribed in ΔABC.

Problem A.7. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio of the perimeter of ΔABC and the radius of the circle circumscribed around ΔABC.

Problem A.8. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio of the length of the circle inscribed in ΔABC to its perimeter.

Problem A.9. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio of the area of the circle inscribed in ΔABC and the area of ΔABC.

Problem A.10. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio of the perimeter of ΔABC and the circumference of the circle inscribed in it.

Problem A.11. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the ratio of area of the circle circumscribed around ΔABC and the area of the circle inscribed in ΔABC.

Problem A.12. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the midcenter G lies on the inscribed circle. Find the tangents of the acute angles in ΔABC.

B. New problems can be created by ***changing their structure*** – the figures that are considered in them are defined to the congruency relation. For this purpose, one metric element must be added to what is given in their statements – whether it is primary or secondary for the triangle or for its constituent parts, into which it is divided by the median CM. At the same time, in this series of problems different elements can be wanted – both metric (lengths of segments, areas of figures), and non-metric (angles or relations of elements of the same dimension). An example of a problem from this series is the following.

Problem B1. In a right triangle ΔABC ($\measuredangle ACB = 90^0$), for which $AC < BC$, the median CM intersects the inscribed circle at point G. If the radius of this circle has length $2\sqrt{3} - 3$, find the length of the hypotenuse and the area of ΔABC.

Note. To find the area of ΔABC, it is not necessary to find the lengths of its legs or the height to the hypotenuse, it is enough to find only the product of the legs.

It is also worth to note that this problem (and also the mentioned problems in series B) can be generalized. For this purpose, the length of the radius of the inscribed circle can be set by a parameter instead of the specific value. Here is its formulation:

Problem B1. (generalized). In a right triangle ΔABC ($\measuredangle ACB = 90^0$), the median CM intersects the inscribed circle at point G. If the length of the radius of this circle is r, find the length of the hypotenuse and the area of ΔABC.

C. New problems can be formulated if the following idea is used. For a right triangle ΔABC ($\measuredangle ACB = 90^0$), in which the median CM intersects the circle inscribed inside it at point G (closer to point M), as a given, let's assume the results (answers) of the twelve above problems (**A.1. - A.12.**) from series **A**, and the conclusion (thesis) of the new problems should be: "Prove that this point G is the midcenter of the given ΔABC ". For this purpose, of course, the specified twelve problems must be solved in advance, in order to use their results as given in the new problems to be created for the proof of the thesis indicated in them. Thus, if this idea is applied to the basic problem, the following new problem is reached.

Problem C1. In a nonisosceles right triangle ΔABC ($\measuredangle ACB = 90^0$), the median CM intersects the circle inscribed inside it at point G (Fig. 1). If the ratio of the radius of the circle inscribed in ΔABC and the radius of the circle described around it is equal to $\frac{4\sqrt{3}-6}{3}$, prove that point G is the midcenter of ΔABC.

The rest of the problems from series **B** are formulated in a similar way.

Conclusions and prospects. The presented methodological development of the considered basic problem enables the current teacher, and even more so the beginner, as well as the students – future teachers of mathematics to master the subtleties of working with planimetric problems. Constructing problems is a creative process that develops various cognitive abilities. The implementation of the last stage of the updated methodological model of the activity of solving mathematical problems (ASMP) is particularly useful, since, on the one hand, it introduces the reader to various ideas for searching and finding a solution, and on the other hand, it presents a methodical technology for constructing new problems and reveals various possibilities for its application. This is extremely valuable for the teacher, because it gives him ideas on how to formulate different problems for individual students or groups of students for an upcoming test, to ensure complete autonomy in solving them by each student. Ensuring independence is important for activating and developing reflexive thinking in the learners, because it contributes to a more conscious mastering of the methods and heuristics used in the implementation of such a complex activity with the considered problems and to improve the methodology of working with them. And all this contributes to increasing the quality of students' mathematical training, helps their intellectual development, to successfully navigate in more complex situations in real reality and ensures a more complete achievement of the goals of mathematics education, which, from its side, leads to building in the modern young person personal qualities necessary for his development throughout his life.

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Boykina D. Geomery problems with an implicitly defined structure – a means for developing intellectual skills in learners.

The article is devoted to planimetric problems related to the midcenter of a triangle and the circle inscribed in it. The significant issue in these problems is, that more specific, additional reasoning is needed to determine their structure, because it is not explicitly define. On one problem, called „basic”, a detailed methodological development is made, having two goals. The first objective is to outline their „educational“ solutions comprehensively, intended to the purposeful preparation of students who will take the matriculation exam in mathematics or candidate student exam, that is why this problem has a criterion character. The other goal is to demonstrate a methodical technology for creating a series of new problems that derive from this basic problem. Different ideas for creating systems of problems are considered, and the corresponding formulations are presented.

***Key words**: methodical development, structure of a problem, methods for solving, intellectual skills, creating of problems.*

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ВПЛИВ ІНТЕРАКТИВНИХ МЕТОДІВ НАВЧАННЯ НА РОЗВИТОК КРИТИЧНОГО МИСЛЕННЯ УЧНІВ ПРИ ВИВЧЕННІ ХІМІЇ

У статті розглядається актуальна проблема розвитку критичного мислення учнів закладів загальної середньої освіти через впровадження інтерактивних методів навчання на уроках хімії. Метою дослідження є теоретичне обґрунтування та експериментальна перевірка системи інтерактивних методів навчання, спрямованих на формування та розвиток навичок критичного мислення здобувачів освіти. Методологія дослідження ґрунтується на комплексному підході, що включає теоретичний аналіз наукової літератури з проблем розвитку критичного мислення, педагогічного експерименту та діагностичних методик. Для оцінки рівня критичного мислення використано стандартизований тест Лорен Старкі, адаптований до українських освітніх реалій. Результати дослідження демонструють значний позитивний ефект від впровадження інтерактивних методів навчання. Зокрема, експериментально доведено підвищення рівня критичного мислення в учнів різних вікових груп, зафіксовано покращення сформованості ключових складові критичного мислення. Практичне значення дослідження полягає в розробці та апробації системи інтерактивних методів